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The Embedding of Proximinal Sets

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A subset M of a normed linear space X is said to be proximinal if $\inf_{m \in M} \|x - m\|$ is attained, for each $x \in X$. If X is embedded in another normed space, Z, a proximinal subset of X may or may not be proximinal in Z. Certain practical problems in multivariate approximation lead us to examine the case when $X = C(S)$ and $Z = C(S \times T)$, where S and T are compact Hausdorff spaces. We characterize the proximinal subspaces of $C(S)$ which are proximinal in $C(S \times T)$ for every T. In another section, a generalization of Mazur's Proximinality Theorem is given. This generalization gives a condition under which a subspace of functions $v \circ f$ is proximinal, when f is held fixed and v ranges over a proximinal subspace.

1. INTRODUCTION

In a normed linear space X, the *distance* from a point x to a (nonempty) subset Y is defined by

$$\operatorname{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\}.$$

If an element y in Y has the property that $\|x - y\| = \operatorname{dist}(x, Y)$, then y is called a *best approximation of x in Y*. If each $x \in X$ has at least one best approximation in Y, then Y is termed *proximinal*. Although proximinal sets are of paramount importance in approximation theory, their structure, behavior, and characteristic properties are very imperfectly understood.

One important question that arises in "global" or "simultaneous"

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approximation is whether a proximinal subset of X is also proximinal as a subset of some larger space Z in which X may be embedded.

PROBLEM I. If X, Y, and Z are Banach spaces satisfying $Y \subset X \subset Z$, and if Y is proximinal in X, under what conditions can we conclude that Y is proximinal in Z?

The study of Problem I is a central theme in this paper. The particular case when $X = C(S)$ and $Z = C(S \times T)$ receives special attention.

Our notation and terminology are standard. Here is a summary. If X is a Banach space, X^* denotes its conjugate — the space of continuous linear functionals on X. If $\varphi \in X^*$, $\ker(\varphi)$ is the kernel or null space of φ. If $Y \subset X$, then $Y^\perp$ is the subspace of X^* composed of all φ satisfying $Y \subset \ker(\varphi)$.

If S and T are sets, and if $f\colon S \to T$, then the *fibres* of f are the subsets of S on which f is constant:

$$f^{-1}[t] = \{s \in S: f(s) = t\}, \qquad \text{for} \quad t \in T.$$

Definitions

Let A and Y be subsets of a Banach space. The following terms are needed.

1. *Chebyshev Radius.* The Chebyshev radius of A relative to Y is the number

$$r_Y(A) = \inf_{y \in Y} \sup_{a \in A} \|a - y\|.$$

If $Y = X$, we write this simply $r(A)$.

2. *Chebyshev Center.* The Chebyshev center of A relative to Y is the set

$$E_Y(A) = \left\{ y \in Y: \sup_{a \in A} \|a - y\| = r_Y(A) \right\}.$$

If $Y = X$, we write this simply $E(A)$.

3. *Property (EK).* The subset Y is said to have property (EK) if $E_Y(A)$ is nonempty for each compact subset A in X.

4. *Property (EO).* The subset Y is said to have property (EO) if $E_Y(A)$ is nonempty for all bounded subsets A in X. This terminology is consistent with [5].

5. *Intervals in $C(S)$.* If S is a topological space, then $C(S)$ denotes the Banach space of all bounded continuous real-valued functions on S. An *interval* in $C(S)$ is a set of the form

$$[u, v] = \{x \in C(S): u \leqslant x \leqslant v\}$$

in which u and v are bounded (but not necessarily continuous) functions on S. We distinguish five types of intervals as follows.

Type I: u and v belong to $C(S)$.

Type II: an interval of type I in which $\sup_s[u(s)-v(s)]=0$.

Type III: u is upper-semicontinuous, v is lower-semicontinuous, and $u \leqslant v$.

Type IV: an interval of type III in which $\sup_s[u(s)-v(s)]=0$.

Type V: an interval of type IV in which $[V, \infty)^- = v$ and $(-\infty, u]^+ = u$ (see Definition 6).

6. *Upper and Lower Envelopes.* If A is a subset of $C(S)$, we put

$$A^+(s) = \inf_{\mathcal{N}} \sup_{\sigma \in \mathcal{N}} \sup_{a \in A} a(\sigma)$$

$$A^-(s) = \sup_{\mathcal{N}} \inf_{\sigma \in \mathcal{N}} \inf_{a \in A} a(\sigma).$$

In these equations, $\mathcal{N}$ runs over all neighborhoods of s. The functions A^+ and A^- are termed the upper and lower envelopes of the set of functions A.

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By examples, it can be shown that in Problem I additional hypotheses are needed for the conclusion that Y is proximinal in Z. In some Banach spaces, an appropriate example can be constructed by taking Y to be a subspace of codimension 2, as in the next result.

THEOREM 1. *Let f and g be two elements of X^* such that $\|\alpha f + \beta g\| = |\alpha| + |\beta|$ for all reals α and β. Assume that for some $z \in \ker(f)$, $g(z) = 1 = \|z\|$. Assume also that $f(x)$ does not attain the value 1 at any point of norm 1 in $\ker(g)$. Then $\ker(f) \cap \ker(g)$ is proximinal in $\ker(f)$ but not in X.*

Proof. Put $Y = \ker(f) \cap \ker(g)$. By the Hahn–Banach theorem

$$\begin{aligned} \operatorname{dist}(x, Y) &= \sup\{\varphi(x) : \varphi \in Y^\perp, \|\varphi\| = 1\} \\ &= \sup\{\alpha f(x) + \beta g(x) : |\alpha| + |\beta| = 1\}. \end{aligned}$$

Fix $x \in \ker(f)$. Then $\operatorname{dist}(x, Y) = |g(x)|$. On the other hand, $x - g(x)z$ is a best approximation to x in Y, as is easily verified. Hence Y is proximinal in $\ker(f)$.

Now fix x in $\ker(g) \backslash \ker(f)$. As in the above calculation, $\operatorname{dist}(x, Y) =$

$|f(x)|$. If $y \in Y$, put $v = (x-y)/f(x)$. Then $f(v) = 1$ and $g(v) = 0$. Consequently $\|v\| > 1$. It follows that

$$\|x - y\| = \|f(x)\, v\| = |f(x)|\; \|v\| > |f(x)| = \text{dist}(x, Y).$$

Hence x has no best approximation in Y. ▮

EXAMPLE 1. Let S be any infinite compact Hausdorff space. Then S contains a sequence of distinct points s_0, s_1, s_2,... such that $s_0 \in F$ and $s_1 \notin F$, where F denotes the closure of $\{s_2, s_3,...\}$. Define functionals by $g(x) = x(s_0)$ and $f(x) = \sum_1^\infty 2^{-n} x(s_n)$. By the Tietze theorem, there is an element $z \in C(S)$ such that $\|z\| = 1$, $z(s_1) = -1$, and $z(s) = 1$ for $s \in F$. It is readily verified that the hypotheses of Proposition 1 are satisfied.

A natural question suggested by Problem I is whether any Banach spaces are "universally" proximinal; i.e., proximinal in every superspace. The finite-dimensional spaces have this property, and, more generally, the reflexive spaces have this property.

POLLUL'S THEOREM. *In order that a Banach space X be proximinal in every Banach space containing X as a subspace, it is necessary and sufficient that X be reflexive.*

This theorem was given first in [8]. See also page 20 in [10].

The next result was proved for $S = [a, b] \subset \mathbb{R}$ by Kadets and Zamyatin [7], for compact S by Zamyatin [13], for paracompact S by Holmes [6, p. 185], and for arbitrary topological spaces S by the present authors [3].

THEOREM 2. *Let S be an arbitrary topological space, and let A be a bounded subset of $C(S)$. Then the Chebyshev center of A is a nonvoid interval of type* IV.

The following result is due to Smith and Ward [11]. See also [3].

THEOREM 3. *Let S be a compact Hausdorff space, Y a proximinal subset of $C(S)$, and A a bounded subset of $C(S)$. In order that the restricted Chebyshev center $E_Y(A)$ be nonvoid it is necessary and sufficient that the function* dist(x, Y) *attain its infimum on the unrestricted Chebyshev center, $E(A)$.*

THEOREM 4. *Let S be a compact Hausdorff space, and Y a subset of $C(S)$. The following are then equivalent.*

(a) *Y has property (EK).*

(b) *Y is proximinal in $C(S \times T)$ for each compact Hausdorff T.*

(c) *Y is proximinal in $C(S \times T)$ for each compact metric T.*

(d) *Y is proximinal in $C(S \times T)$ for each compact T in $C(S)$.*

Proof. Assume (a), and let T be any compact Hausdorff space. Let $z \in C(S \times T)$ and $y \in Y$. Then

$$\|y - z\| = \sup_t \sup_s |y(s) - z(s, t)| = \sup_t \|y - z^t\| \tag{1}$$

where $z^t(s) = z(s, t)$. Since $\{z^t : t \in T\}$ is a compact subset of $C(S)$, there exists a $y \in Y$ which minimizes (1) on Y. Hence (b) is true.

It is obvious that (b) implies (c) and that (c) implies (d). In order to prove that (d) implies (a), assume (d) and let T be a compact set in $C(S)$. Define $z \in C(S \times T)$ by the equation $z(s, t) = t(s)$ for $t \in T$ and $s \in S$. By (d), z has a best approximation in Y. By (1) this element will belong to the Chebyshev center of T relative to Y. ∎

A similar proof establishes the next result.

THEOREM 5. *Let S be a compact Hausdorff space, and Y a subset of $C(S)$. The following are equivalent.*

(a) *Y has property* (EO).

(b) *Y is proximinal in $C(S \times T)$ for every topological space T.*

(c) *Y is proximinal in $C(S \times T)$ for every metric space T.*

(d) *Y is proximinal in $C(S \times T)$ for every $T \subset C(S)$.*

EXAMPLE. Let Y be any proximinal subspace of finite codimension in $C(S)$, S being compact Hausdorff. Then, by a theorem of Garkavi [4], Y has property (EK). Hence it has the other properties in Theorem 4.

Open Problem. Do there exist proximinal subspaces of $C(S)$ which do not have property (EK)?

The following important theorem, established by Smith and Ward [12], provides many examples to illustrate Theorem 5.

THEOREM 6. *If S is compact Hausdorff, then every closed subalgebra of $C(S)$ has the property* (EO).

For a far-reaching generalization of this, see the recent paper of Amir, Mach, and Saatkamp [1].

We recall a little of the theory of Chebyshev centers in a space $C(S)$, S compact Hausdorff. If A is a bounded subset of $C(S)$, one defines A^+ and

A^- as in Section 1. Then the Chebyshev center of A is the interval $[A^+ - r, A^- + r]$, where r is defined by

$$r = \tfrac{1}{2}\|A^+ - A^-\|.$$

This number is also the "Chebyshev radius" of A.

THEOREM 7. *Let S be a compact Hausdorff space. An interval in $C(S)$ is the Chebyshev center of a compact set in $C(S)$ if and only if it is an interval of type* II.

Proof. If $[u, v] = E(A)$ then $u = A^+ - r$ and $v = A^- + r$ with $r = \frac{1}{2}\|A^+ - A^-\|$. If s is a point where $r = \frac{1}{2}[A^+(s) - A^-(s)]$ then $u(s) - v(s) = 0$.
Conversely, suppose that $[u, v]$ is of Type II. Define $\rho = \frac{1}{2}\|v - u\|$ and $A = \{v - \rho, u + \rho\}$. Then one can verify easily that $E(A) = [u, v]$. ∎

The following lemmas are elementary or well known. See [9], especially pages 100 and 98.

LEMMA 1. *If S is a normal space and if $[u, v]$ is an interval of type* III, *then* $\sup\{x(s) : x \in [u, v]\} = v(s)$.

LEMMA 2. *Let a, b, and c be bounded functions on a topological space S. If $[a, b]$ is nonempty and $c < a$ then*

$$\sup\{x(s) : x \in [a, b]\} = \sup\{x(s) : x \in [c, b]\}.$$

LEMMA 3. *Let $[u, v]$ be an interval of type* III *in $C(S)$ with S a normal space. Let F be a closed subset of S, and let $x \in C(F)$, with $u|F \leqslant x \leqslant v|F$. Then x has an extension in $[u, v]$.*

THEOREM 8. *Let S be a compact Hausdorff space. In order that an interval in $C(S)$ be the Chebyshev center of set in $C(S)$ it is necessary and sufficient that it be an interval of type* V.

Proof. Necessity. Assume that $[u, v] = E(A)$ for some $A \subset C(S)$. By the theory of centers, $u = A^+ - r$ and $v = A^- + r$, where r is the Chebyshev radius of A. By the definition of r, $\|a - x\| \leqslant r$ whenever $a \in A$ and $x \in E(A)$. From this we conclude that $x - r \leqslant a \leqslant x + r$. Taking a supremum and infimum for $x \in E(A) = [u, v]$ we have

$$\sup x(s) - r \leqslant a(s) \leqslant \inf x(s) + r.$$

By Lemma 1, this yields $v - r \leqslant a \leqslant u + r$. This establishes that $A \subset [v - r, u + r]$, whence $u + r = A^+ \leqslant [v - r, u + r]^+ \leqslant u + r$. Thus $[v - r, u + r]^+ =$

$u+r$. By Lemma 2, $(-\infty, u+r]^+ = u+r$, and so $(-\infty, u]^+ = u$. Similarly, $[v, \infty)^- = v$.

Sufficiency. Suppose that the interval $[u, v]$ is of type V. Select ρ so that the set $D=[v-\rho, u+\rho]$ is nonempty. Then a straightforward calculation using Lemma 2 shows that $D^+ = u+\rho$ and $D^- = v-\rho$. From the theory of centers, the Chebyshev radius of D is

$$\tfrac{1}{2}\max[D^+(s)-D^-(s)] = \tfrac{1}{2}\max[2\rho + u(s) - v(s)] = \rho$$

and the center of D is $[D^+ - \rho, D^- + \rho] = [u, v]$. ∎

EXAMPLE. An interval $[u, v]$ of type IV which is not of type V is given by $v(s)=1$, $0 \leqslant s \leqslant 1$; $u(s)=0$, $0 \leqslant s < 1$; $u(1)=1$.

THEOREM 9. *Let S be an infinite compact metric space. Than $C(S)$ contains a proximinal hyperplane which is not proximinal in $C(S \times T)$ for some (noncompact) bounded metric space T.*

Proof. Select distinct points $s_1, s_2,...$ in S and define the functional $\varphi \in C(S)^*$ by the equation

$$\varphi(x) = \sum_{n=1}^{\infty} 2^{-n} x(s_n).$$

Let $Y=\ker(\varphi)$. Since φ attains its supremum on the unit cell of $C(S)$, Y is proximinal. By the theorem of Zamyatin cited below, Y does not have property (EO). Hence there exists a bounded set $T \subset C(S)$ such that Y is not proximinal in $C(S \times T)$. ∎

ZAMYATIN'S THEOREM. *Let S be a compact metric space. A subspace Y of finite codimension in $C(S)$ has property* (EO) *if and only if each element of $Y^\perp$ has finite support.*

THEOREM 10. *Let S be a compact Hausdorff space and let Y be a proximinal subset of $C(S)$. In order that Y be proximinal in $C(S \times T)$ for every compact Hausdorff space T it is necessary and sufficient that* $\operatorname{dist}(x, Y)$ *attain its infimum on every interval of type* II. *In order that Y be proximinal in $C(S \times T)$ for every topology space T it is necessary and sufficient that* $\operatorname{dist}(x, Y)$ *attain its infimum on every interval of type* V.

Proof. Assume that Y is proximinal in $C(S \times T)$ for all compact Hausdorff T. Let $[u, v]$ be an interval of type II. By Theorem 7, $[u, v] = E(A)$ for a compact set A in $C(S)$. By Theorem 4, $E_Y(A)$ is nonvoid. By Theorem 3, dist (x, Y) attains its infimum on $[u, v]$.

Assume that dist(x, Y) attains its infimum on each interval of type II. By Theorem 7, dist(x, Y) attains its infimum on $E(A)$ for any compact A. By Theorem 3, $E_Y(A)$ is nonempty for any compact A. Hence Y has property (EK). By Theorem 4, Y is proximinal in $C(S \times T)$ for every compact Hausdorff space T.

The other assertion of the theorem is proved in the same way using Theorems 3, 5, and 8. ∎

As an illustration of the preceding theorem, we give a short proof of a known result (in fact, a corollary to Theorem 6.)

THEOREM 11. *Each ideal in $C(S)$ has property* (EO).

Proof. Let $Y = \{x \in C(S) \colon x|F = 0\}$ where F is a closed subset of S. Let $[u, v]$ be any interval of type III. By Theorem 3.3 of [3], the interval $[u|F, v|F]$ is proximinal in $C(F)$. Select $w \in [u|F, v|F]$ of minimal norm. By Lemma 3, w has an extension w^* in $[u, v]$. For any $z \in [u, v]$ we have dist$(z, Y) = \|z|F\| \geqslant \|w^*|F\| =$ dist(w^*, Y). Thus dist(x, Y) attains its infimum on each interval of type III. By Theorems 10 and 5, Y has property (EO). ∎

DEFINITION. If Y is a subspace of $C(S)$ such that the function dist(x, Y) attains its infimum on each interval of type IV, then Y is said to have property (EV).

REMARK. We note the following implications for a subspace of $C(S)$:

$$(\mathrm{EV}) \Rightarrow (\mathrm{EO}) \Rightarrow (\mathrm{EK}) \Rightarrow (\mathrm{E}).$$

Here (E) denotes ordinary proximinality. Zamyatin's theorem shows that (EK) does not imply (EO). Theorem 15 (below) shows that (EO) does not imply (EV).

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This section is devoted to the construction of some new proximinal subspaces in $C(S)$. These are the form

$$Z = \{v \circ f \colon v \in V\}$$

where $f \colon S \to T$ is a continuous map of the compact space S onto the compact space T, and V is a subspace of $C(T)$. The special case when $V = C(T)$ is covered by a theorem of Mazur, conveniently accessible in Semadeni's treatise [9, p. 124].

THEOREM 12. *Assume that the subspace V referred to above is proximinal in $C(T)$ and has property* (EV). *Then the subspace Z is proximinal in $C(S)$.*

Proof. Define $Y = \{u \circ f \colon u \in C(T)\}$. Let x be an arbitrary element of $C(S)$. Define, for $t \in T$,

$$x'(t) = \sup\{x(s) \colon f(s) = t\}$$
$$x''(t) = \inf\{x(s) \colon f(s) = t\}$$
$$\rho = \tfrac{1}{2}\|x' - x''\|.$$

On page 124 of [9] one can find the proofs of the following facts.

(1) x' is upper-semicontinuous on T.

(2) x'' is lower-semicontinuous on T.

(3) $\operatorname{dist}(x, Y) = \rho$.

(4) $P_Y(x) = \{u \circ f \colon u \in C(T) \text{ and } x' - \rho \leqslant u \leqslant x'' + \rho\}$.

Here $P_Y(x)$ denotes the set of all best approximations to x in the subspace Y.

Let $J = [x' - \rho, x'' + \rho]$. Then J is an interval of type IV in $C(T)$. Note that J is usually not a Chebyshev center. By hypothesis, there is an element $\bar{u} \in J$ such that

$$\operatorname{dist}(\bar{u}, V) = \inf_{u \in J} \operatorname{dist}(u, V).$$

Since V is proximinal, there is an element $\bar{v} \in V$ such that

$$\|\bar{u} - \bar{v}\| = \operatorname{dist}(\bar{u}, V).$$

Hence

$$\|\bar{u} - \bar{v}\| = \inf_{u \in J} \inf_{v \in V} \|u - v\|.$$

Since f is surjective, we can conclude that

$$\begin{aligned}\|\bar{u} \circ f - \bar{v} \circ f\| &= \inf_{u \in J} \inf_{v \in V} \|u \circ f - v \circ f\| \\ &= \operatorname{dist}(P_Y(x), Z).\end{aligned}$$

Now it follows that

$$\begin{aligned}\|x - \bar{v} \circ f\| &\leqslant \|x - \bar{u} \circ f\| + \|\bar{u} \circ f - \bar{v} \circ f\| \\ &= \operatorname{dist}(x, Y) + \operatorname{dist}(P_Y(x), Z).\end{aligned}$$

By Lemma 4, which follows,

$$\|x-\bar{v}\circ f\| = \operatorname{dist}(x, Z). \quad \blacksquare$$

Remark. If the infimum of $\operatorname{dist}(u, V)$ is not attained on $[x'-\rho, x''+\rho]$, then x has no best approximation in Z.

LEMMA 4. *For any x and V, we have*

$$\operatorname{dist}(x, Z) = \operatorname{dist}(x, Y) + \operatorname{dist}(P_Y(x), Z).$$

Proof. We adopt all the notation of the preceding theorem and its proof. For any $v \in C(T)$ we have

$$\begin{aligned}
\|x - v\circ f\| &= \sup_s |x(s) - v(f(s))| = \sup_t \sup_{f(s)=t} |x(s)-v(t)| \\
&= \sup_t \sup_{f(s)=t} \max\{x(s)-v(t), v(t)-x(s)\} \\
&= \sup_t \max\{x'(t)-v(t), v(t)-x''(t)\} \\
&= \sup_t \max\{x'(t)-v(t), v(t)-x'(t), v(t)-x''(t), x''(t)-v(t)\} \\
&= \sup_t \max\{|x'(t)-v(t)|, |x''(t)-v(t)|\} \\
&= \max\{\|x'-v\|, \|x''-v\|\}.
\end{aligned}$$

In the midst of this calculation, we used the elementary inequalities $x' \geqslant x''$, $v-x'' \geqslant v-x'$, and $x'-v \geqslant x''-v$. Now use Theorem 1 of [2, p. 70]. The result is

$$\|x-v\circ f\| = \|w + |c-v|\,\|$$

in which $w = \frac{1}{2}(x'-x'')$ and $c = \frac{1}{2}(x'+x'')$. By the same technique used in the proof of Theorem 3.7 in [3], the previous equation is transformed to

$$\|x-v\circ f\| = \|w\| + \inf_{u\in J} \|u-v\|.$$

Since f is surjective, this becomes

$$\|x-v\circ f\| = \|w\| + \inf_{u\in J} \|u\circ f - v\circ f\|.$$

Hence

$$\inf_{v\in V} \|x-v\circ f\| = \|w\| + \inf_{v\in V} \inf_{u\in J} \|u\circ f - v\circ f\|$$

or, in other terms,

$$\operatorname{dist}(x, Z) = \operatorname{dist}(x, Y) + \operatorname{dist}(P_Y(x), Z). \quad \blacksquare$$

Remark. As this proof shows, the approximation of x by an element of Z can be interpreted as the simultaneous approximation problem for the semicontinuous functions x' and x'' in $V \subset C(T)$.

THEOREM 13. *Let f, V, and Z be as described at the beginning of this section. Assume that f is an open map and that V has property* (EK). *Then Z is proximinal in $C(S)$.*

Proof. Since V has property (EK), it has all the properties listed in Theorem 4. By Theorem 10, $\operatorname{dist}(u, V)$ attains its infimum on each interval of type II in $C(T)$. Now the intervals $J = [x' - \rho, x'' + \rho]$ which occur in the proof of Theorem 12 are actually of type II by the following Lemma. As the proof of Theorem 12 then shows, Z is proximinal. $\blacksquare$

LEMMA 5. *If f is an open map then for each $x \in C(S)$, x' and x'' belong to $C(T)$.*

Proof. We know that x' is upper-semicontinuous. In order to prove it lower-semicontinuous, let $\alpha \in \mathbb{R}$ and put $A = \{t \in T\colon x'(t) > \alpha\}$. It is to be shown that A is open.

Let $t_0 \in A$. Since $x'(t_0) > \alpha$, there is a point $s_0 \in f^{-1}[t_0]$ such that $x(s_0) > \alpha$. Let $\mathcal{O} = \{s \in S\colon x(s) > \alpha\}$. Since f is an open map, $f[\mathcal{O}]$ is open; it is a neighborhood of t_0. If $t \in f[\mathcal{O}]$ then $t = f(s)$ for some $s \in \mathcal{O}$. Hence $x'(t) \geqslant x(s) > \alpha$, so $t \in A$. Thus $f[\mathcal{O}] \subset A$. $\blacksquare$

EXAMPLE. Let $S = [-1, 1]$, $T = [0, 1]$, $f(s) = s^2$. Then f is open, the map $t \to f^{-1}[t]$ is lower-semicontinuous, and for each $x \in C(S)$, x' and x'' are continuous.

EXAMPLE. Let $S = [0, 2\pi]$, $T = [-1, 1]$, $f(s) = \cos ns$. Then f is open.

THEOREM 14. *Let $S = T = [0, 1]$. There exists an f having finite fibres and a proximinal hyperplane V in $C(T)$ such that Z is not proximinal.*

Proof. Let $V = \ker \varphi$, with $\varphi(u) = \int_0^1 u(t)\,dt$. Define $f\colon S \to T$ by

$$\begin{aligned} f(s) &= 2s && (0 \leqslant s \leqslant \tfrac{1}{3}) \\ &= 1 - s && (\tfrac{1}{3} \leqslant s \leqslant \tfrac{2}{3}) \\ &= 2s - 1 && (\tfrac{2}{3} \leqslant s \leqslant 1). \end{aligned}$$

Let $x(s) = s + \frac{1}{4}$. Then x has no best approximation in Z. By referring to Theorem 12 and its proof, we see that in order to prove this assertion it suffices to verify that the function $\operatorname{dist}(u, V)$ does not attain its infimum on $[x' - \rho, x'' + \rho]$. Now $\operatorname{dist}(u, V) = |\varphi(u)|$, and a calculation reveals that

$$\begin{aligned} x'(t) - \rho &= t/2 && (0 \leqslant t < \tfrac{1}{3}) \\ &= (t+1)/2 && (\tfrac{1}{3} \leqslant t \leqslant 1) \\ x''(t) + \rho &= (t+1)/2 && (0 \leqslant t < \tfrac{2}{3}) \\ &= 1 + t/2 && (2/3 \leqslant t \leqslant 1). \end{aligned}$$

The functional $|\varphi(u)|$ does not attain its infimum on the interval in question. ∎

We are indebted to Dan Amir for the following theorem.

THEOREM 15. *There exists a subspace Y in a space $C(S)$ such that Y has property* (EO) *but not property* (EV).

Proof. Let $S = \beta\mathbb{N}$, the Stone–Čech compactification of the positive integers. Define $\varphi \in l_\infty^*$ by the equation $\varphi(x) = \sum_1^\infty 2^{-n} x(n)$. The support of φ is the closure of $\mathbb{N}$ in $\beta\mathbb{N}$, and is therefore $\beta\mathbb{N}$. Hence the support of φ is extremally disconnected. By a theorem in [5], $Y = \ker(\varphi)$ has property (EO). In order to show that Y does not have property (EV), let $u(s)$ be the characteristic function of $\beta\mathbb{N} \setminus \mathbb{N}$. Since this set is closed, u is upper-semicontinuous. Let $v(s) = 1$. Now, $\operatorname{dist}(x, Y) = |\varphi(x)|$. This function does not attain its infimum on the interval $[u, v]$ because if $x \in [u, v]$ then $x(n)$ is eventually 1. ∎

THEOREM 16. *Let S be a compact metric space, and Y a subspace of finite codimension in $C(S)$ having the property* (EO). *Then the function* $\operatorname{dist}(x, Y)$ *attains its infimum on any interval of type* III.

Proof. By Zamyatin's theorem, each element of Y has finite support. Since Y has finite codimension, the set $K = \bigcup\{\operatorname{supp}(\varphi) \colon \varphi \in Y^\perp\}$ is finite. Then

$$\begin{aligned} (1) \quad \operatorname{dist}(x, Y) &= \max\{\varphi(x) \colon \varphi \in Y^\perp, \|\varphi\| = 1\} \\ &= \operatorname{dist}(x|K, Y|K). \end{aligned}$$

Let $[u, v]$ be an interval of type III in $C(S)$. Then

$$(2) \quad \{x \in C(K) \colon u|K \leqslant x \leqslant v|K\}$$

is compact. Since the distance function in (1) is continuous, it attains its infimum at some point x_0 in the set (2). By Lemma 3, x_0 has an extension in $[u, v]$. This extension has minimal distance to Y in $[u, v]$. ∎

We use this opportunity to point out an error in our earlier paper [3]. Theorem 4.2 is incorrect. It was based upon a misunderstanding of Zamyatin's theorem. In [14] Zamyatin proved that if T is a compact *metric* space, then the subspaces of finite codimension in $C(T)$ which have property (EO) are those whose annihilating functionals have finite support. In [5], Garkavi and Zamyatin proved that if T is an arbitrary compact Hausdorff space, then the proximinal subspaces of finite codimension having the EO property are those whose annihilating functionals have extremally disconnected supports (in the relative topology of the support).

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